ON THE INTEGRATION OF THE EQUATIONS OF UNSTEADY CREEP OF SOLID BODIES

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1. We shall consider a body made of a material subject to creep; let it occupy the volume V, bounded by the surface S. On part S_1 of S let there be given the stresses

$$\sigma_x \cos nx + \tau_{xy} \cos ny + \tau_{xz} \cos nz = f_x \qquad (xyz) \tag{1.1}$$

on part S_2 the components of the velocity vector

$$v_x = v_x^*, \qquad v_y = v_y^*, \qquad v_z = v_z^* \tag{1.2}$$

Inside the body one has the equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \qquad (xyz) \tag{1.3}$$

Here and in what follows the symbol (xyz) denotes that the unwritten formulas or expressions of components are to be determined by cyclic permutation of the letters x, y, z.

The surface loading and body forces may, generally speaking, depend on time t. We will assume that the components of the strain-rates

$$\xi_x = \frac{\partial v_x}{\partial x}, \ \ldots, \ \eta_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}, \ \ldots \qquad (xyz) \tag{1.4}$$

are determined by the equations of the theory of creep of Kachanov [1]

$$\xi_x = \frac{1}{E}\left[\frac{\partial \sigma_x}{\partial t} - \nu\left(\frac{\partial \sigma_y}{\partial t} + \frac{\partial \sigma_z}{\partial t}\right)\right] + \xi_x^c, \qquad \eta_{xy} = \frac{1}{G}\frac{\partial \tau_{xy}}{\partial t} + \eta_{xy}^c \qquad (xyz) \tag{1.5}$$

where the components of creep ξ_x^c, η_{xy}^c, ... are known functions of the stresses and time t:

$$\xi_x^c = F(T, t)(\sigma_x - \sigma), \qquad \eta_{xy}^c = 2F(T, t)\tau_{xy} \tag{1.6}$$

(T is the intensity of the shear stresses).

The initial elastic state of stress of the body (for $t = 0$) is assumed to be known:

$$\sigma_x = \sigma_x^\circ (x, y, z), \ldots, \tau_{xy} = \tau_{xy}^\circ (x, y, z) \qquad (xyz) \tag{1.7}$$

2. The considered interval of time will be subdivided by the points $t = 0$, $t = t_1$, ..., $t = t_i$, ... into small segments Δt (which, generally speaking, may not be equal to each other).

Differentiating with respect to time the static boundary conditions (1.1) and the equilibrium equations (1.3), we obtain

$$\frac{\partial \sigma_x}{\partial t} \cos nx + \frac{\partial \tau_{xy}}{\partial t} \cos ny + \frac{\partial \tau_{xz}}{\partial t} \cos nz = \frac{\partial f_x}{\partial t} \tag{2.1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \sigma_x}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \tau_{xy}}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \tau_{xz}}{\partial t}\right) + \frac{\partial F_x}{\partial t} = 0 \qquad (xyz) \tag{2.2}$$

In the equations (1.4), (1.5), (2.1) and (2.2) we set $t = t_i$ and replace the time derivatives by finite differences

$$\left.\frac{\partial \sigma_x}{\partial t}\right|_{t=t_i} = \frac{\Delta_i \sigma_x}{\Delta t} \qquad (xyz)$$

We then obtain the following system of equations for the stress and strain increments:

$$\frac{\partial \Delta \sigma_x}{\partial x} + \frac{\partial \Delta \tau_{xy}}{\partial y} + \frac{\partial \Delta \tau_{xz}}{\partial z} + \Delta F_x = 0 \qquad (xyz) \tag{2.3}$$

$$\Delta \varepsilon_x = \frac{1}{E}[\Delta \sigma_x - \nu(\Delta \sigma_y + \Delta \sigma_z)] + \delta_x, \qquad \Delta \gamma_{xy} = \frac{1}{G}\Delta \tau_{xy} + \delta_{xy} \qquad (xyz) \tag{2.4}$$

Here

$$\Delta F_x = \left(\frac{\partial F_x}{\partial t}\right)_{t=t_i} \Delta t, \qquad \Delta \varepsilon_x = (\xi_x)_{t=t_i}\Delta t, \qquad \delta_x = (\xi_x^c)_{t=t_i}\Delta t,$$

$$\delta_{xy} = (\eta_{xy}^c)_{t=t_i}\Delta t \qquad \text{etc.} \tag{2.5}$$

The boundary conditions (2.1), (1.2) give

$$\Delta \sigma_x \cos nx + \Delta \tau_{xy} \cos ny + \Delta \tau_{xz} \cos nz = \Delta f_x \quad \text{or } S_1 \qquad (xyz) \tag{2.6}$$

$$\Delta u_x = \Delta u_x^* \quad \text{or } S_2 \tag{2.7}$$

where $\Delta u_x = v_x|_{t=t_i}\Delta t$, $\Delta u_x^* = v_x^*|_{t=t_i}\Delta t$ etc. are the increments of the displacement components in the body and on the surface in time Δt. Obviously

$$\Delta \varepsilon_x = \frac{\partial \Delta u_x}{\partial x}, \ldots, \Delta \gamma_{xy} = \frac{\partial \Delta u_x}{\partial y} + \frac{\partial \Delta u_y}{\partial x} \qquad (xyz) \tag{2.8}$$

Thus, the determination of the increments of stresses and deformations in the time Δt are reduced to a peculiar linear problem which in many respects is analogous to a problem of thermoelasticity. The difference from the last consists only of the fact that the given additional deformations $\delta_x, \ldots, \delta_{xy}$ calculated in accordance with (1.6), (2.5) from

$$\delta_x = [F(T, t)(\sigma_x - \sigma)]\,\Delta t, \quad \delta_{xy} = 2\,[F(T, t)\,\tau_{xy}]\,\Delta t \qquad \text{for } t = t_i, \quad \sigma_x = \sigma_{xi}, \ldots \quad (xyz) \tag{2.9}$$

are present in (2.4) in the expressions for the extensions as well as for the shears, and, generally speaking, $\delta_x \neq \delta_y \neq \delta_z$. Using (2.9) we note that the supplementary deformations satisfy the condition of incompressibility

$$\delta_x + \delta_y + \delta_z = 0 \tag{2.10}$$

It is not difficult to demonstrate (for example, by constructing with the aid of (2.3), (2.4), (2.6), (2.8) and (2.10) the equilibrium equations in terms of displacements) that the determination of the displacements $\Delta u_x, \Delta u_y$ and Δu_z in the problem under consideration reduces to a traditional isothermal problem of the theory of elasticity with the additional loadings

$$\delta f_x = \frac{E}{1+\nu}\left[\delta_x \cos nx + \tfrac{1}{2}(\delta_{xy}\cos ny + \delta_{xz}\cos nz)\right] \quad (xyz) \tag{2.11}$$

on the surface S_1 and

$$\delta F_x = -\frac{E}{1+\nu}\left[\frac{\partial \delta_x}{\partial x} + \frac{1}{2}\left(\frac{\partial \delta_{xy}}{\partial y} + \frac{\partial \delta_{xz}}{\partial z}\right)\right] \quad (xyz) \tag{2.12}$$

throughout the body.

The analysis of unsteady creep of a body thus reduces to the evaluation of increments in the stresses and deformations for consecutive small time intervals Δt. The additional loads at each stage must be calculated from the formulas (2.9), (2.11) and (2.12) using the results of the evaluation of the preceding integral. The initial values for this process are given by the solution (1.7).

In the case of a uniform state of stress (for example, in the problem of stress relaxation of a rod) the stated numerical process reduces to the numerical integration of the original equations by the method of Euler [2].

We note that the description of the numerical procedure does not depend essentially on the actual form of the formulas (1.6) and therefore one may apply it also to other theories of creep, for example, the theory of hardening.

3. The solution of the "pseudo-thermal" problem of the theory of elasticity to which the integration of the equations of creep were reduced may be constructed for example on the basis of (2.11) and (2.12) if the Green function of the corresponding elastic problem is known. In

a number of cases it is not difficult to obtain the required general solution based directly on the original equations.

We shall consider, as an example, the unsteady creep of a twisted circular rod of radius a.

By (2.4) one has

$$\Delta\gamma_{\varphi z} = \frac{1}{G}\,\Delta\tau_{\varphi z} + \delta(r) \tag{3.1}$$

where $\delta(r) = \delta_{\phi z}$ must be assumed to be an arbitrary function of the radius r. Setting, as usually, $\Delta\gamma_{\phi z} = r\,\Delta\theta$ and taking into consideration that the external torque is constant, we find from the condition of static equilibrium

$$\Delta\tau_{\varphi z} = G\left[\frac{4r}{a^2}\int_0^a \delta(r)\,r^2 dr - \delta(r)\right] \tag{3.2}$$

4. We note that the studied numerical process may be generalized to the case of presence of plastic deformations. We will start from the equations of the theory of plastic flow

$$d\varepsilon_x{}^p = \Phi(T)(\sigma_x - \sigma)\,dT, \qquad d\gamma_{xy}{}^p = 2\Phi(T)\,\tau_{xy}dT \quad (xyz) \text{ for } T = T_m,\ dT \geqslant 0 \tag{4.1}$$

and

$$d\varepsilon_x{}^p = d\varepsilon_y{}^p = \ldots = d\gamma_{xy}{}^p = 0 \qquad \text{for } T \leqslant T_m \text{ ior } T = T_m, \text{ but } dT \leqslant 0. \tag{4.2}$$

Here I_m is the maximum value of the intensity T, attained during the entire loading history.

Supplementing (2.4) by finite increments of plastic deformation, calculated from (4.1), we obtain for the loading stage

$$\Delta\varepsilon_x = c_{11}\Delta\sigma_x + c_{12}\Delta\sigma_y + \ldots + c_{16}\Delta\tau_{xy} + \delta_x \qquad (xyz) \tag{4.3}$$

where the coefficients c_{ik} determine the state of stress at the beginning of the time interval Δt under consideration.

The instantaneous position of the boundary of the region of unloading is determined by the condition $\Delta t = 0$. In the region of unloading Equation (2.4) will apply. The relation (4.3) represents the law of deformation of a certain anisotropic body.

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